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A Quantum Theory primer – algebraic perspective

Observables and states

In physics, we have to do with objects (systems,...), characterized by sets of physical quantities which may be measured in well defined ways, denoted by $\{A', B', ...\}$. These objects may be prepared in various ways, denote the set of possible 'preparations' by $\{\omega', \eta', ...\}$.

Admit from the beginning, even in the classical case, presence of uncertainty in the description. Then a specific preparation of a system ω' is characterized by probability measures $P^{A'}_{\omega'}(\Omega)$ on Borel sets in $\mathbb{R}$ for all physical quantities (determining the results of repeated measurements). Now let

$$\omega'_1 \sim \omega'_2 \qquad \Longleftrightarrow \qquad P^{A'}_{\omega'_1}(\Omega) = P^{A'}_{\omega'_2}(\Omega) \quad \text{for all } A' \text{ and } \Omega\,.$$

Then a **state** of a system is an equivalence class $\omega \equiv [\omega']_\sim$. Next, let

$$A'_1 \sim A'_2 \qquad \Longleftrightarrow \qquad P^{A'_1}_{\omega}(\Omega) = P^{A'_2}_{\omega}(\Omega) \quad \text{for all } \omega \;\text{ and } \Omega\,.$$

Then an **observable** is an equivalence class $A \equiv [A']_\sim$. In this way we have a set of probability measures $P^A_\omega(\Omega)$ on $\mathbb{R}$.

Each continuous function $f : \mathbb{R} \mapsto \mathbb{R}$ is assumed to define for each observable A its deformation $f(A)$ with

$$P^{f(A)}_\omega(\Omega) = P^A_\omega(f^{-1}(\Omega))\,.$$

Now we take a slightly new look: a state ω will be now identified with the expectation value

$$\omega(A) \equiv \int_{\mathbb{R}} \lambda\, dP^A_\omega(\lambda)\,.$$

This new interpretation of a state is legitimate, as then for all continuous functions f:

$$\omega(f(A)) = \int_{\mathbb{R}} \mu\, dP^{f(A)}_\omega(\mu) = \int_{\mathbb{R}} f(\lambda)\, dP^A_\omega(\lambda)\,,$$

and $P^A_\omega(\Omega)$ may be recovered from ω (take $0 \leq f_n \nearrow \chi_\Omega$ – the characteristic function of Ω, then $P^A_\omega(\Omega) = \lim \omega(f_n(A))$).

A physical theory must now clarify what structure behind the scene determines correlations between probability distributions of various observables in a given state (or, equivalently, between expectation values of functions of different observables).

Classical theory

Let M be a topological space (say, a 'phase space'). We assume it is a compact (for simplicity – e.g. obtained by some compactification) Hausdorff space.

Observables: functions $A \in C_{\mathbb{R}}(M)$ continuous on M. If we equip M with a regular Borel measure μ, such that $\mu(M) = 1$, then (M, μ) is a probability space and observables are stochastic variables.

States: determined by $\mu \mapsto \omega_\mu$,

$$P^A_{\omega_\mu}(\Omega) = \mu(A^{-1}(\Omega)) .$$

Then

$$\omega_\mu(A) = \int_{\mathbb{R}} \lambda \, dP^A_{\omega_\mu}(\lambda) = \int_M A(m) \, d\mu(m) ,$$

and the general relations for $\omega_\mu(f(A))$ are satisfied.

But here more general constructions are possible: if $A_1, \ldots, A_n$ are observables and $f : \mathbb{R}^n \mapsto \mathbb{R}$ is continuous then $f(A_1, \ldots, A_n)$ is an observable, and the relation

$$\begin{aligned} \omega_\mu(f(A_1, \ldots, A_n)) &= \int_M f(A_1(m), \ldots, A_n(m)) \, d\mu(m) \\ &= \int_{\mathbb{R}^n} f(\lambda_1, \ldots, \lambda_n) \, dP^{A_1, \ldots, A_n}_{\omega_\mu}(\lambda_1, \ldots, \lambda_n) \end{aligned}$$

determines **joint probability distributions** $P^{A_1, \ldots, A_n}_{\omega_\mu}(\Omega_1 \times \ldots \times \Omega_n)$. This is characterized by saying that *all classical observables are comeasurable.*

Now we want to look at these constructions from a different angle.

It will be convenient to complexify $C_{\mathbb{R}}(M)$ and consider all complex continuous functions $C(M)$. This is a complex vector space, and with the norm $\|A\| = \sup_{m \in M} |A(m)|$ – a Banach space. Observables are now the real elements of this space. For each probability measure μ the corresponding state ω_μ, extended to $C(M)$ by

$$\omega_\mu(A) = \int_M A(m) \, d\mu(m)$$

has the following properties:

1. is a bounded linear functional, thus $\omega_\mu \in C(M)^*$, the dual space,
2. satisfies $\|\omega_\mu\| = 1 = \omega_\mu(\mathbf{1})$ (where $\mathbf{1}(m) = 1$ for all m),
3. is positive, that is $\omega_\mu(A) \geq 0$ for $A \geq 0$ (that is $A(m) \geq 0$ for all m).

But now conversely, if ω satisfies 1–3 (in fact, not all of these assumptions are independent) then the Riesz representation theorem says that ω uniquely determines a probability measure (regular Borel measure) such that $\omega = \omega_\mu$ in the above sense. Thus from now on a **state is a functional with 1–3**.

Let ω_i, $i = 1, 2$ be states. Then for $\lambda, (1-\lambda) \geq 0$ the functional $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$ is also a state: states form a convex subset of $C(M)^*$. Conversely, if ω has a decomposition as above, then it is evident that ω_i are finer states than ω (look at the corresponding decomposition of measures). We say that ω is **pure** if it has no such nontrivial $(\lambda, (1-\lambda) > 0)$ decomposition. It is evident that the only pure states in the present construction are those connected with measures concentrated at a point, thus given by

$$\omega_m(A) = A(m)\,, \quad \text{so also} \quad \omega_m(f(A)) = f(A(m))\,.$$

Therefore, in these states values of all observables are strictly determined. Moreover, for each probability measure μ one has the unique decomposition

$$\omega_\mu = \int_M \omega_m \, d\mu(m)\,.$$

This is summarized by saying that **in classical physics probability is not fundamental – it can be removed in principle** (although very often not in practice).

To modify this scheme so as to adapt it to the Quantum Theory we need to identify some more structure.

C*-algebras

Suppose that a complex linear space $\mathcal{A}$ is equipped with the structures:

(i) $\mathcal{A}$ is an algebra,

(ii) $\mathcal{A}$ is a Banach space,

(iii) $\|AB\| \leq \|A\|\|B\|$ for all $A, B \in \mathcal{A}$

then $\mathcal{A}$ is called a **Banach algebra**. If in addition $\mathcal{A}$ is equipped with involution $A \mapsto A^*$ (i.e. $(\alpha A + \beta B)^* = \bar{\alpha}A^* + \bar{\beta}B^*$, $(AB)^* = B^*A^*$ and $(A^*)^* = A$) with which it satisfies

(iv) $\|A^*A\| = \|A\|^2$ (which together with (iii) implies $\|A^*\| = \|A\|$)

then it is called a **C*-algebra**. If the algebra $\mathcal{A}$ has an identity element $\mathbf{1}$, it is called C*-algebra with identity. We assume from now on that $\mathcal{A}$ has an identity (typical case in applications; one can always adjoin an identity element). What is special about C*-algebras? First of all:

Examples

1. $\mathcal{B}(\mathcal{H})$ – the algebra of bounded operators on a Hilbert space $\mathcal{H}$ – is a C*-algebra
2. each norm-closed subalgebra of $\mathcal{B}(\mathcal{H})$ is a C*-algebra
3. $C(M)$ (as above) is a *commutative* C*-algebra.

And now more generally.

Thm. *Let $\pi : \mathcal{A} \mapsto \mathcal{B}$ be a *-morphism between two C*-algebras, i.e.*

$$\pi(\alpha A + \beta B) = \alpha\pi(A) + \beta\pi(B)\,, \quad \pi(AB) = \pi(A)\pi(B)\,, \quad \pi(A^*) = \pi(A)^*\,,$$

Then $\pi(\mathcal{A})$ is a C-algebra and $\|\pi(A)\| \leq \|A\|$. Therefore if π is a *-isomorphism then*

$$\|\pi(A)\| = \|A\| \quad \textit{for all} \;\; A \in \mathcal{A}.$$

We can now characterize C*-algebras by:

Thm. *(Gelfand-Naimark) Each C*-algebra is *-isomorphic to a norm-closed subalgebra of bounded operators on a Hilbert space.*

We shall say that a *-morphism $\pi : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H})$ is a **representation** of $\mathcal{A}$, which is **faithful** if it is injective (thus $\pi : \mathcal{A} \mapsto \pi(\mathcal{A})$ is a *-isomorphism).

We call $A \in \mathcal{A}$ **positive**, if it can be written as $A = B^*B$, $B \in \mathcal{A}$; a linear functional ω on $\mathcal{A}$ is called **positive** if $\omega(A) \geq 0$ for all positive A.

Prop. *Positive functional ω satisfies:*

$$\omega(A^*) = \overline{\omega(A)}\,, \qquad |\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B)\,.$$

Proof is immediate by $\omega((A + \lambda B)^*(A + \lambda B)) \geq 0$ for all $\lambda \in \mathbb{C}$.

Let ω be positive. If $\|A\| < 1$ then by power expansion one finds in the algebra $(\mathbf{1} - A)^{1/2}$; if in addition $A^* = A$ then $\mathbf{1} - A = D^2$, $D^* = D$, thus $\mathbf{1} - A$ is positive. Therefore for $C^* = C$ there is $|\omega(C)| \leq \omega(\mathbf{1})(\|C\| + \varepsilon)$ for all $\varepsilon > 0$, so $|\omega(C)| \leq \omega(\mathbf{1})\|C\|$. Putting $A = \mathbf{1}$ in the proposition we thus find

$$|\omega(B)|^2 \leq \omega(\mathbf{1})\omega(B^*B) \leq [\omega(\mathbf{1})\|B\|]^2\,,$$

so ω is bounded with $\|\omega\| = \omega(\mathbf{1})$.

A **state** on the algebra $\mathcal{A}$ is a positive linear functional with $\omega(\mathbf{1}) = 1$ (automatically in $\mathcal{A}^*$). The set of states $\Sigma(\mathcal{A})$ is a convex set.

Quantum theory

Looking back we can now interpret the formalism for classical theory in this way: observables of a classical system form a commutative C*-algebra $C(M)$ (strictly speaking observables are selfadjoint elements in the algebra, but one extends the name to the whole of algebra) and physical states are the states on the algebra (in the above algebraic sense).

Quantum theory is obtained by abandoning the commutativity assumption. One postulates: **A specific quantum theory is formed by a choice of a non-abelian C*-algebra; physical states of the theory are positive, normalized functionals on the algebra**.

It is not yet clear at this stage how to obtain a probabilistic interpretation for this case. To obtain it we have to develop mathematics.

Spectrum and states

For $A \in \mathcal{A}$ the resolvent set $r_{\mathcal{A}}(A)$ is the set of complex numbers λ for which $\lambda\mathbf{1} - A$ has an inverse in $\mathcal{A}$, the complement is the **spectrum** $\sigma_{\mathcal{A}}(A)$.

One can show that if $A \in \mathcal{B} \subseteq \mathcal{A}$ then $\sigma_{\mathcal{A}}(A) = \sigma_{\mathcal{B}}(A)$. In fact, one shows that $\lambda\mathbf{1} - A$ is invertible iff it is invertible in the algebra generated by $\mathbf{1}, A, A^*$, so one writes $r(A)$, $\sigma(A)$. For $\lambda \in r(A)$: $(\lambda\mathbf{1} - A)^{-1}$ is called the **resolvent**.

Thm. *Spectrum $\sigma(A)$ has the following properties:*

1. *it is closed*
2. *is contained in the closed disk of radius $\|A\|$*
3. $\sigma(A^*) = \overline{\sigma(A)}$
4. $\sigma(P(A)) = P(\sigma(A))$, *$P$ – polynomial*
5. *for $A^* = A$: $\sigma(A) \subseteq \langle -\|A\|, +\|A\|\rangle$; for positive A: $\sigma(A) \subseteq \langle 0, \|A\|\rangle$.*

Moreover, for $A^* = A$ the algebra generated by $\mathbf{1}, A$ is isomorphic to the algebra $C(\sigma(A))$ of complex continuous functions on the spectrum.

Therefore, by Riesz representation theorem, for a state ω and each observable $A^* = A$ there is a unique probability measure P^A_ω such that

$$\omega(f(A)) = \int_{\sigma(A)} f(\lambda)\, dP^A_\omega(\lambda)\,.$$

Statistical interpretation for each single observable is then as in the classical case. However, it is not possible, in general, to form new (selfadjoint) observables as functions of more than one observable, and the structure of the set of states needs to be investigated anew.

Take two selfadjoint observables A, B and denote $AB + BA = D$ and the commutator $[A, B] = AB - BA = iC$. Then C, D are also selfadjoint observables, and

$$\omega(C^2) \leq |\omega(BAC)| + |\omega(ABC)| \leq \sqrt{\omega(B^2)\omega(CA^2C)} + \sqrt{\omega(A^2)\omega(CB^2C)} \\ \leq (\sqrt{\omega(B^2)}\|A\| + \sqrt{\omega(A^2)}\|B\|)\sqrt{\omega(C^2)}\,,$$

where we used the fact that $A \mapsto \omega(CAC)$ is a positive linear functional, therefore $|\omega(CA^2C)| \leq \|A\|^2\omega(CC)$ above. The commutator is unchanged under the translation $A \to A - \alpha\mathbf{1}$, $B \to B - \beta\mathbf{1}$, so we have

$$\sqrt{\omega(C^2)} \leq \sqrt{\omega((B-\beta\mathbf{1})^2)}\|A - \alpha\mathbf{1}\| + \sqrt{\omega((A-\alpha\mathbf{1})^2)}\|B - \beta\mathbf{1}\|\,.$$

Now, the probability measure P^A_ω is concentrated in one point α iff $\omega((A - \alpha\mathbf{1})^2) = 0$, and in general infimum of this quantity over α is a measure of the width of the distribution. Thus if P^A_ω and P^B_ω are point measures, then P^C_ω is a point measure concentrated in 0. More generally, if the widths of A and B are small in this state, then the width of C must be small. In particular, if $0 \notin \sigma(C)$ then there is nonzero lower bound for the rhs (spectrum is closed).

Therefore, in general there are no joint probability distributions, in contrast to the classical case. Nevertheless, we can ask what states are 'best possible', i.e. pure.

Representations and states

Let $\pi : \mathcal{A} \mapsto \mathcal{B}(\mathcal{H})$ be a representation and $\psi \in \mathcal{H}$ any normalized vector. Then it is easily seen that ω_ψ defined by

$$\omega_\psi(A) = (\psi, \pi(A)\psi) = \mathrm{Tr}[\pi(A)P_\psi],$$

where P_ψ is the orthogonal projection operator onto one-dimensional subspace spanned by ψ, is a state on $\mathcal{A}$, called a **vector state**. Forming finite convex combinations of such states one obtains states of the form:

$$\omega_\rho(A) = \mathrm{Tr}[\pi(A)\rho],$$

where ρ is positive, finite-rank operator with $\mathrm{Tr}\,\rho = 1$. As $|\omega_\rho(A)| \leq \|\rho\|_{\mathrm{Tr}}\,\|\pi(A)\| = \|\pi(A)\|$, the above definition may be extended in trace-norm to all positive, trace-class operators ρ with $\mathrm{Tr}\,\rho = 1$. This class of states is called **the folium of** π or **states normal with respect to** π.

Interpretation:
A C^*-algebra defines a specific physical theory in the general frame of quantum theory. A choice of representation π gives a specific physical setting (system) which obeys this theory, and the set of normal states – possible states of this concrete system. Different representations, i.e. not unitarily equivalent, usually describe non-comparable systems – systems differing infinitely, in some sense. (Unitarily equivalent representations differ only by a 'choice of coordinates'). For example, states of finite amount of matter, and thermodynamic limit states, differ infinitely and belong to different folia.

Each state ω_ρ with ρ of rank greater than 1 is mixed (not pure). Are vector states pure? And more generally, if we are given a state abstractly as a functional on an algebra, can we connect it with some representation, and when is it pure?

The answer comes in the two following results. We shall denote by $(\pi, \mathcal{H}, \Omega)$ a cyclic representation π with a normalized, cyclic vector $\Omega \in \mathcal{H}$ (i.e. linear hull of $\pi(A)\Omega$ is dense in $\mathcal{H}$).

Thm. (GNS construction) *For each state ω on $\mathcal{A}$ there is a unique, up to unitary equivalence, representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$, such that*

$$\omega(A) = (\Omega_\omega, \pi_\omega(A)\Omega_\omega)$$

for all $A \in \mathcal{A}$ (i.e. ω is the vector state defined by Ω_ω).

Sketch of a proof
Given ω define on $\mathcal{A}$ a positive semi-definite scalar product $\langle A, B\rangle = \omega(A^*B)$. By Schwarz inequality for ω the set $\mathcal{I} \subseteq \mathcal{A}$ formed by elements for which $\langle A, A\rangle = 0$ is a subspace, and left ideal in $\mathcal{A}$. From the first property it follows that $\mathcal{A}/\mathcal{I}$ is a pre-Hilbert space. On this space define: $\pi_\omega(A)[B] = [AB]$. As $\mathcal{I}$ is a left ideal this is a consistent definition, and has all algebraic properties of a representation on $\mathcal{A}/\mathcal{I}$. Finally, $\|[AB]\|^2 = \omega(B^*A^*AB) \leq \|A\|^2\|[B]\|^2$, so $\|\pi_\omega(A)[B]\| \leq \|A\|\|[B]\|$ and π_ω extends to the completion $\mathcal{H}_\omega$ of $\mathcal{A}/\mathcal{I}$, with cyclic vector $\Omega_\omega = [\mathbf{1}]$.
If $(\pi'_\omega, \mathcal{H}'_\omega, \Omega'_\omega)$ also satisfies the condition of the thesis, then $U\pi_\omega(A)\Omega_\omega = \pi'_\omega(A)\Omega'_\omega$ defines the unitary equivalence.

Recall that a set of operators acts irreducibly in a Hilbert space iff each vector in this space is cyclic for this set. Thus in an irreducible representation π all GNS representations of vector states are equivalent to π.

Thm. *State ω over $\mathcal{A}$ is pure iff the corresponding GNS representation $(\pi_\omega, \mathcal{H}_\omega, \Omega_\omega)$ is irreducible.*

Proof. If $\pi_\omega = \pi_1 \oplus \pi_2$, then $\Omega_\omega = \sqrt{\lambda}\,\Omega_1 \oplus \sqrt{1-\lambda}\,\Omega_2$ and $\omega = \lambda\omega_1 + (1-\lambda)\omega_2$, where $\omega_i(A) = (\Omega_i, \pi_i(A)\Omega_i)$. Conversely, if ω has decomposition as above and π_i are GNS representations for ω_i, then representation on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by

$$\pi(A)[\sqrt{\lambda}\,\Omega_1 \oplus \sqrt{1-\lambda}\,\Omega_2] = \sqrt{\lambda}\,\pi_1(A)\Omega_1 \oplus \sqrt{1-\lambda}\,\pi_2(A)\Omega_2$$

is unitarily equivalent to GNS representation of ω.

As a corollary we obtain: *in an irreducible representation* $(\pi, \mathcal{H})$ *of* $\mathcal{A}$ *all vector states in* $\mathcal{H}$ *are pure.*

Example

Weyl algebra on $\mathbb{R}^2$:
For all $\xi = (a, b) \in \mathbb{R}^2$ let $W(\xi)$ be elements generating a complex *-algebra by relations

$$W(\xi_1)W(\xi_2) = \exp[-\tfrac{i}{2}\{\xi_1, \xi_2\}]\, W(\xi_1 + \xi_2)\,, \quad W(\xi)^* = W(-\xi)\,, \quad W(0) = \mathbf{1}\,,$$

where $\{(a_1, b_1), (a_2, b_2)\} = a_1 b_2 - b_1 a_2$ is a symplectic form on $\mathbb{R}^2$.

Thm. *This algebra extends to a unique (up to an isomorphism) C*-algebra.*

For each ξ the elements $W(\lambda\xi)$, $\lambda \in \mathbb{R}$, form a one-parameter group of unitary elements. Representation π is called regular if $\lambda \mapsto \pi(W(\lambda\xi))$ is weakly continuous for each ξ. In this case by Stone thm $\pi(W(\lambda\xi)) = \exp[-i\lambda G(\xi)]$, where $G(\xi)$ are selfadjoint operators on the representation Hilbert space.

Thm. *All irreducible, regular representations of this Weyl algebra are unitarily equivalent to the one acting on* $L^2(\mathbb{R})$ *by* $G(1, 0) = X$, $G(0, 1) = P$, *where*

$$[X\psi](x) = x\psi(x)\,, \qquad [P\psi](x) = -i\frac{d}{dx}\psi(x)$$

– *the Schrödinger representation.*

Strictly speaking, X and P, being unbounded operators, are defined on some dense subspaces. On the Schwartz functions space $\mathcal{S}$ they satisfy 'canonical commutation relations': $[X, P] = i\,\mathrm{id}$.

This system has the interpretation of one particle with position observable X and momentum P. There is only one realization of such system.

Here is a good moment to say why is the quantum structure not observed in everyday life. Operators X and P in the above formulation are physically dimensionless (scaled by some dimensional constants). If one restores physical dimensions then it is evident that a constant of physical dimension (length)x(mass)x(length/time) is needed on the rhs of commutation relation. It turns out that this is a new fundamental constant

$$\hbar \simeq 1.054 \times 10^{-27} \mathrm{g\,cm^2/sec}\,, \qquad [X, P] = i\hbar\,\mathrm{id}\,.$$

All algebraic relations in C*-algebras of QT have this property, that commutators $[A, B]$ vanish if one sets $\hbar = 0$. As $\hbar$ is an extremely small, in everyday measures, quantity, one does not see quantum effects directly.

Co-measurability

Let us go back to the question of co-measurability of observables. We already know that if $[A, B] \neq 0$ then only in exceptional cases some of the values of A and B may take simultaneously sharp values.

Suppose, then, that $[A, B] = 0$; then also for any representation π there is $[\pi(A), \pi(B)] = 0$. Spectral theorem in Hilbert space for commuting selfadjoint operators tells us, that there are spectral families $\{P_\Omega\}$, $\{Q_\Omega\}$, $\Omega \subseteq \mathbb{R}$, such that

$$\pi(A) = \int_{\sigma(A)} \lambda \, dP_\lambda \,, \quad \pi(B) = \int_{\sigma(B)} \lambda \, dQ_\lambda \,, \quad [P_\Omega, Q_{\Omega'}] = 0 \,.$$

For each density operator ρ defining a state in this representation the function $\Omega \times \Omega' \mapsto \mathrm{Tr}(\rho P_\Omega Q_{\Omega'})$ defines a probability measure on $\sigma(A) \times \sigma(B)$ – joint probability distribution of A and B.

Statistical independence and entanglement

Suppose that $\mathcal{A}_1$ and $\mathcal{A}_2$ are independent C*-algebras. One can form the tensor product of algebras $\mathcal{A}_1 \otimes \mathcal{A}_2$ and obtains a *-algebra. Now take any faithful representations π_1 and π_2 of $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively and introduce the norm on $\mathcal{A}_1 \otimes \mathcal{A}_2$ as the operator norm on $\pi_1(\mathcal{A}_1) \otimes \pi_2(\mathcal{A}_2)$. It turns out that this norm is independent of the choice of faithful representations and has the C*-property. The C*-algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ is the theory of systems consisting of independent subsystems governed by $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively.

To see this take any representations π_1 and π_2 and states in these representations described by density operators ρ_1 and ρ_2. The product operator $\rho = \rho_1 \otimes \rho_2$ is a density operator defining a state in the product representation. In this situation for $A_i \in \mathcal{A}_i$ the observables $A_1 \otimes \mathbf{1}$ and $\mathbf{1} \otimes A_2$ are not only comeasurable, but also their joint probability distributions in product states are statistically independent, that is the joint probability measure is the product of probability measures for each of the observables.

However, the are other states which are not of the product form. Take a vector state $\psi = \alpha\varphi_1 \otimes \varphi_2 + \beta\chi_1 \otimes \chi_2$. If π_i are irreducible then this is a pure state (thus 'maximum information state'). However, joint probability distribution in this state is not a product measure any more, so there are correlations between outcomes of measurements of the two observables. This is the much celebrated in recent time effect of 'entanglement'.

Field theory example

In quantum statistical mechanics and in quantum field theory one has to do with systems with 'infinite degrees of freedom'. Precise sense of this statement is this: the C*-algebra behind them is a Weyl algebra built in analogy to the above construction, but in which vectors ξ are, instead of $\mathbb{R}^2$, elements of an infinite-dimensional real vector space $\mathcal{L}$ equipped with a nondegenerate symplectic form $\{\xi_1, \xi_2\}$. (This characterization does not include fermionic systems, which we do not discuss here.)

While in this more general case the theorem on uniqueness of the generated C*-algebra remains true, the unitary equivalence of regular representations does not hold.

Example
Take $\mathcal{D} = \mathcal{D}_{\mathbb{R}}(\mathcal{M})$, the space of smooth real functions of compact support on Minkowski space (real four-dimensional affine space with scalar product of signature $(+,-,-,-)$ on the associated vector space). Each scalar field on $\mathcal{M}$ which satisfies the Klein-Gordon equation

$$(\Box + m^2)\phi(x) = 0$$

and has initial data of compact support on a Cauchy surface may be represented by

$$\phi(x) = \int_{\mathcal{M}} \Delta(x-y) f(y)\, dy$$

where $f \in \mathcal{D}$, and Δ is the distribution solving the initial conditions problem for the Klein-Gordon equation: if Σ is the $t = 0$ hyperplane (t – time) and denote points in $\mathcal{M}$ by $(t, \vec{x})$ then

$$\phi(t, \vec{x}) = \int_{\Sigma} \big\{ \Delta(t, \vec{x} - \vec{y})\chi(\vec{y}) - \partial_t \Delta(t, \vec{x} - \vec{y})\rho(\vec{y}) \big\}\, d^3y$$

satisfies the K-G equation and the initial conditions

$$\phi(0, \vec{x}) = \rho(\vec{x})\,, \quad \partial_t \phi(t, \vec{x})_{|t=0} = \chi(\vec{x})\,.$$

Let $\mathcal{D}_0$ be the subspace of $\mathcal{D}$ consisting of those functions f for which the corresponding field ϕ is zero field, and denote $\mathcal{L} = \mathcal{D}/\mathcal{D}_0$, the quotient space. Define on $\mathcal{D}$ a symplectic form by

$$\{f, g\} = \int_{\mathcal{M}\times\mathcal{M}} f(x)\Delta(x-y)g(y)\, dx\, dy = -\{g, f\}\,, \quad \text{as} \quad \Delta(-x) = -\Delta(x)\,.$$

This form is consistently transferred to $\mathcal{L}$ by

$$\{[f], [g]\} = \{f, g\}\,,$$

and on this space it is nondegenerate. The Weyl algebra over $(\mathcal{L}, \{.,.\})$ forms the system of observables connected with systems of identical spinless particles with mass m. The observable $W([f])$ may be measured in the region of $\mathcal{M}$ given by supp$f$.

In a regular representation of this algebra one has $\pi(W([tf])) = \exp[-it\Phi_\pi(f)]$, and the unbounded generators – field operators – sytisfy commutation relations

$$[\Phi_\pi(f), \Phi_\pi(g)] = i\{f, g\}\,.$$

This quantum field satisfies K-G equation in the weak sense: $\Phi_\pi((\Box + m^2)f) = 0$.

Symmetries

Physical theories usually have some symmetries: they are unchanged when subjected to some transformations representing such physical operations as: time or space translation, rotations, Lorentz transformations (linear isometries of the Minkowski product).

Let $\mathcal{A}$ be the algebra of observables of some theory. Recall that strictly speaking observables are selfadjoint elements of $\mathcal{A}$, that is elements of the real subspace of $\mathcal{A}_\mathbb{R}$ invariant under conjugation. A minimal demand on a symmetry is that it defines a bijective mapping $\alpha_\mathbb{R} : \mathcal{A}_\mathbb{R} \mapsto \mathcal{A}_\mathbb{R}$ which conserves the structure of operations in that space which form observables from observables. Therefore, in particular, $\alpha_\mathbb{R}$ is a real-linear automorphism of $\mathcal{A}_\mathbb{R}$, so it can uniquely be extended to a linear automorphism $\alpha : \mathcal{A} \mapsto \mathcal{A}$ by $\alpha(A+iB) = \alpha_\mathbb{R}(A) + i\alpha_\mathbb{R}(B)$. It follows that

$$\alpha(A^*) = \alpha(A)^*\,.$$

Further, as for $A, B \in \mathcal{A}_\mathbb{R}$ also $\{A, B\} \equiv AB + BA$ and $[A, B]^2$ are in $\mathcal{A}_\mathbb{R}$, so we demand

$$\alpha_\mathbb{R}(\{A, B\}) = \{\alpha_\mathbb{R}(A), \alpha_\mathbb{R}(B)\}\,, \quad \alpha_\mathbb{R}(A^2) = \alpha_\mathbb{R}(A)^2\,, \quad \alpha_\mathbb{R}([A, B]^2) = [\alpha_\mathbb{R}(A), \alpha_\mathbb{R}(B)]^2$$

(in fact, only the first condition turns out to be independent). But $i[A, B] \in \mathcal{A}_\mathbb{R}$, so from these equations we have

$$\alpha_\mathbb{R}(i[A, B])^2 = \alpha_\mathbb{R}\big((i[A, B])^2\big) = -\alpha_\mathbb{R}\big([A, B]^2\big) = \big(i[\alpha_\mathbb{R}(A), \alpha_\mathbb{R}(B)]\big)^2\,.$$

There are two obvious possibilities to satisfy this demand. Assume that

$$\alpha_\mathbb{R}(i[A, B]) = \pm i[\alpha_\mathbb{R}(A), \alpha_\mathbb{R}(B)]\,,$$

where the sign on the rhs is common for all $A, B \in \mathcal{A}_\mathbb{R}$. (This exhausts the needs for physical applications; also, a general case turns out to be some superposition of these two cases) If we write $AB = \frac{1}{2}\{A, B\} - \frac{i}{2}\, i[A, B]$, use the definition of α and the above conditions then

$$\alpha(AB) = \tfrac{1}{2}\{\alpha_\mathbb{R}(A), \alpha_\mathbb{R}(B)\}) \pm \tfrac{1}{2}[\alpha_\mathbb{R}(A), \alpha_\mathbb{R}(B)] = \begin{cases} \alpha(A)\alpha(B) \\ \alpha(B)\alpha(A) \end{cases}\,,$$

which extends to all $A, B \in \mathcal{A}$.

We summarize:
Physical symmetries are described by linear automorphisms or linear anti-automorphisms of the C-algebra of observables.*

Let a physical theory be encoded in the algebra $\mathcal{A}$ and a symmetry of this theory be described by an automorphism α. Let further a concrete physical system obeying this theory be described by representation π of the algebra. The mapping $\pi \circ \alpha$ defines another representation of the algebra. The symmetry α of the theory is actually realized in the system considered if the representations π and $\pi \circ \alpha$ are unitarily equivalent: there is a unitary operator U such that

$$\pi(\alpha(A)) = U\pi(A)U^*$$

for all A. If such U does not exist, the **symmetry is broken**.

In case α is an anti-automorphism the implementability condition is modified as follows: there is an anti-unitary operator K such that

$$\pi(\alpha(A)) = K\pi(A)^*K^*, .$$

Symmetry groups

Physical symmetries very often form groups. Let G be a group of symmetries of a theory defined by the C*-algebra $\mathcal{A}$, so each element g in the group defines an automorphism α_g of $\mathcal{A}$. In this case, however, there is additional structure: successive application of the transformations connected with g_1 and g_2 should be the same as application of the transformation connected with $g_2 g_1$. Therefore the mapping $g \mapsto \alpha_g$ from G into the group of automorphisms of $\mathcal{A}$ is a homomorphism:

$$\alpha_{g_2 g_1} = \alpha_{g_2}\alpha_{g_1} .$$

(If for each $g \in G$ there is an $h \in G$ such that $g = h^2$, which is true e.g. for Lie groups, then this shows that α could not be replaced by an anti-automorphism, and we restrict to such situation). Suppose that the symmetries are actually implemented in the representation π by unitary operators U_g. Then it follows that

$$U_{g_2 g_1}\pi(A)U^*_{g_2 g_1} = U_{g_2}U_{g_1}\pi(A)U^*_{g_1}U^*_{g_2}$$

for all A. Equivalently: $U_{g_2 g_1}(U_{g_2}U_{g_1})^*$ commutes with all $\pi(A)$. Suppose that π is irreducible; then

$$U_{g_2}U_{g_1} = \omega(g_2, g_1)U_{g_2 g_1} ,$$

where $\omega(g_2, g_1)$ is a complex number of unit module. Thus U_g is a unitary projective representation of G.

Let G be a Lie group. Then for each one parameter subgroup $g(t)g(s) = g(t + s)$ one can modify phase factors in front of each $U_{g(t)}$ so as to obtain: $U'_{g(t)}U'_{g(s)} = U'_{g(t+s)}$, and therefore by Stone theorem $U'_{g(t)} = \exp[-itH]$, where H is a (usually unbounded) selfadjoint operator – generator of this one parameter symmetry. For $g(t)$ – time translation the generator is the energy operator, and for space translation it is a component of the momentum operator.

Example
For the Weyl algebra over $\mathbb{R}^2$ introduced earlier the transformations ($t \in \mathbb{R}$)

$$\alpha_t(W((a,b)) = W((a, b + at/m))$$

extend to automorphisms of the algebra. Moreover, $\alpha_t\alpha_s = \alpha_{t+s}$. These automorphisms induce in the Schrödinger representation transformations

$$X \mapsto X + \frac{t}{m}P\,, \qquad P \mapsto P\,,$$

– time evolution of a free particle with mass m. It is easy to show that the unitary one-parameter group

$$U(t) = \exp[-itP^2/2m]$$

implements these transformations, thus $H = \frac{1}{m}P^2$ is the energy of the particle.

Quantum Field Theory

The example of the Weyl algebra for a particle shows all fundamental features of non-relativistic quantum mechanics. In Galilean spacetime time is a distinguished coordinate transforming under Galilean transformations onto itself. The structure of QM respects this fact. Algebra of observables is to be understood to describe observables at any given, fixed time. This has interpretational consequences: (i) observables are assumed to be measurable instantaneously; (ii) time, in contrast to position, is a parameter, not an observable.

When constructing a quantum theory in relativistic spacetime (Minkowski space) one is confronted with the fact that space and time coordinates mix under spacetime geometric symmetries (Poincre transformations). Therefore, on the one hand one abandons the nonrealistic idea of instantaneous measurements and assumes only that observables correspond to measurements taking some finite stretch of time. On the other hand, in agreement with the mixing of space and time, one adds some localization of measurements in space directions. In this way rather than upgrading time to an observable, one deprives space coordinates of this status.

In this way one arrives at a mapping $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O}) \subseteq \mathcal{A}$, which ascribes subalgebras $\mathcal{A}(\mathcal{O})$ to regions in which observables in these subalgebras are measured. The field Weyl algebra discussed above is an example of this structure.

Let $\mathcal{P}$ be the Poincaré group of spacetime affine isometries, that is $(a,\Lambda)(x) = \Lambda x + a$, where Λ is a Lorentz transformation. This Poincare group should also be a symmetry of a relativistic theory. Let us see it on the example of scalar Weyl algebra.

Poincare group acts on $\mathcal{D}$ by a representation

$$(a,\Lambda) \mapsto T_{(a,\Lambda)}\,, \quad \big[T_{(a,\Lambda)} f\big](x) = f(\Lambda^{-1}(x-a))\,.$$

As the distribution Δ is Lorentz-invariant, $\Delta(\Lambda x) = \Delta(x)$, the representation T is formed of symplectic mappings:

$$\{T_{(a,\Lambda)} f, T_{(a,\Lambda)} g\} = \{f,g\}\,.$$

Therefore the mappings

$$\alpha_{(a,\Lambda)}(W([f]) = W([T_{(a,\Lambda)} f])$$

define a symmetry group of $\mathcal{A}$.

One now looks for a representation in which this symmetry is implemented by $U_{(a,\Lambda)}$, has a vector invariant under these U ('vacuum') and cyclic for the representation (all vector states may be generated by the action of the algebra on vacuum). In fact, it is sufficient to find a state (as a positive functional) invariant under α.

Thm. *If ω is a state on $\mathcal{A}$ invariant under an automorphism α, that is $\omega(\alpha(A)) = \omega(A)$, then α is implemented in the corresponding GNS representation by*

$$U\pi_\omega(A)\Omega_\omega = \pi_\omega(\alpha(A))\Omega_\omega\,.$$

If $G \ni g \mapsto \alpha_g$ is a group homomorphism then U_g is a representation of G.

Suppose now that in the setting of the Weyl algebra as above we have found a state invariant under Poincare group and the corresponding representation $U_{(a,\Lambda)}$. If we restrict the group to translations then

$$U_{(a,\mathbf{1})}U_{(b,\mathbf{1})} = U_{(a+b,\mathbf{1})}\,.$$

Suppose the dependence on a is continuous, then by Stone theorem there exist selfadjoint commuting four-momentum operators P^μ such that $U_{(a,\mathbf{1})} = \exp[-ia\cdot P]$. As the four-momentum operators commute they have a joint spectrum. Physical interpretation imposes now a further restriction: as the vacuum should be the lowest energy state in each Minkowski reference system, the spectral vectors p^μ should all lie in the forward lightcone.

One can show that this demand finally fixes the vacuum state uniquely. The four-momentum spectrum has then an isolated point $p^\mu = 0$, a discrete hyperboloid $p\cdot p = m^2$, $p^0 > 0$ and continuous spectrum covering the set $p\cdot p \geq 4m^2$, $p^0 > 0$. The state with $p = 0$ is the vacuum, the states with spectral content restricted to the hyperboloid are one-particle states, and those with spectral content restricted to continuous spectrum – many particle states.

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