Asymptotic structure of electrodynamics revisited

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The lecture presents:

- Overview of classical asymptotic structure of interacting electrodynamics:
  - Invariant measure on null directions
  - Free electromagnetic fields:
    representation, selection criteria, null asymptotes
  - Observability of infrared degrees and "memory effect"
  - Matter radiation system, timelike infinity
  - Spacelike tail and "matching property"
  - Dirac field, timelike infinity
  - Invariant structures and Poincaré transformations
  - "Large gauge transformation"
- A quantization scheme of this structure:
  - Quantum algebra
  - Representations
  - Quantum variables at spacial infinity

invariant measure on null directions

- future lightcone

$$C_+ = \{ l \mid l \cdot l = 0,\ l^0 > 0 \} .$$

- t – any future-pointing timelike, unit vector

$$C_+^t = \{ l \in C_+ \mid t \cdot l = 1 \}$$

- $f(l)$ – such that for each $\gamma > 0$ there is $f(\gamma l) = \gamma^{-2} f(l)$
  $d\Omega_t(l)$ –the angle measure on the unit sphere

$$\int f(l)\, d^2 l \stackrel{\text{def}}{=} \int_{C_+^t} f(l)\, d\Omega_t(l) \qquad t - \text{independent}$$

- generators of Lorentz transformation, intrinsic differential operators on the lightcone

$$L_{ab} \stackrel{\text{def}}{=} l_a \frac{\partial}{\partial l^b} - l_b \frac{\partial}{\partial l^a}$$

then

$$\int L_{ab} f(l)\, d^2 l = 0 .$$

homogeneous Maxwell equations

- Fourier representation

$$A(x) = \frac{1}{\pi} \int e^{-ix\cdot k} a(k) \,\mathrm{sgn}(k^0) \delta(k^2) d^4 k \,,$$

- reality, Lorenz condition and gauge transformation:

$$\overline{a(k)} = -a(-k)\,, \quad k \cdot a(k) = 0\,, \quad a(k) \to a(k) + k\beta(k)$$

- selection first step: finite energy-momentum

$$P^a = -\int \overline{a(k)} \cdot a(k)\, k^a \, d\mu_0(k)\,.$$

- selection second step: Coulomb-like decay

$$A^{\mathrm{as}}(y) = \lim_{\lambda\to\infty} \lambda A(\lambda y)\,, \qquad y^2 < 0$$

$$a^{\mathrm{as}}(k) = \lim_{\mu\searrow 0} \mu\, a(\mu k)\,;$$

- $\omega \, \mathrm{Re}\, a(\omega l)$ – continuous
  $\omega \, \mathrm{Im}\, a(\omega l)$ – jumps by $2 \, \mathrm{Im}\, a^{\mathrm{as}}(l)$ at $\omega = 0$
- "conservation law" $(y^2 < 0)$

$$\lim_{\lambda \to \infty} \lambda A(x + \lambda y) = \frac{1}{\pi} \int \frac{\mathrm{Im}\, a^{\mathrm{as}}(l)}{y \cdot l} \, d^2 l + \int \mathrm{Re}\, a^{\mathrm{as}}(l) \, \delta(y \cdot l) \, d^2 l$$

- radiation potentials produced in scattering –
  – long-range tails are even in y

integral representation

- denote ($s \in \mathbb{R}$, $l \in C_+$, overdot$= \partial/\partial s$)

$$\dot{V}(s,l) = -\int_{-\infty}^{+\infty} \omega a(\omega l) e^{-i\omega s}\, d\omega\,,$$

$$\dot{V}(\mu s, \mu l) = \mu^{-2}\dot{V}(s,l) \quad (\mu > 0)\,, \qquad l \cdot \dot{V}(s,l) = 0\,.$$

- representation

$$A(x) = -\frac{1}{2\pi}\int \dot{V}(x \cdot l, l)\, d^2 l\,,$$

- gauge

$$\dot{V}(s,l) \to \dot{V}(s,l) + l\dot{\alpha}(s,l)\,.$$

- asymptotics

$$\dot{V}(s,l) = -\frac{2}{s}\,\mathrm{Im}\, a^{\mathrm{as}}(l) + O(|s|^{-1-\epsilon}) \quad \text{for} \quad |s| \to \infty\,.$$

null asymptotes

- selection third step: well-behaved null asymptotics:

$$\operatorname{Im} a^{\mathrm{as}}(l) = 0$$

- choose $V(+\infty, l) = 0$
- denote $V'(s,l) = -V(s,l) + V(-\infty, l)$, so $V'(-\infty, l) = 0$
- then

$$V(-\infty, l) = V'(+\infty, l) = 2\pi \lim_{\omega \to 0} \omega a(\omega l) = 2\pi a^{\mathrm{as}}(l)$$

$$\lim_{R\to\infty} RA_b(x + Rl) = V_b(x \cdot l, l)\,,$$

$$\lim_{R\to\infty} RA_b(x - Rl) = V'_b(x \cdot l, l)\,,$$

$$\lim_{R\to\infty} RF_{ab}(x + Rl) = l_a \dot{V}_b(x \cdot l, l) - l_b \dot{V}_a(x \cdot l, l)\,,$$

$$\lim_{R\to\infty} RF_{ab}(x - Rl) = l_a \dot{V}'_b(x \cdot l, l) - l_b \dot{V}'_a(x \cdot l, l)\,.$$

no magnetic monopoles

- selection fourth step: spacial tail of electrical type
- then
$$L_{[ab}V_{c]}(-\infty, l) = L_{[ab}V'_{c]}(+\infty, l) = 0$$
- it follows
$$l_a V_b(-\infty, l) - l_b V_a(-\infty, l) = L_{ab}\Phi(l)$$
$$\Phi(l) = \frac{1}{4\pi}\int \frac{l \cdot V(-\infty, l')}{l \cdot l'}\, d^2 l'$$
- gauge transformation
$$V(-\infty, l) \to V(-\infty, l) + l\alpha(-\infty, l)\,,$$
$$\Phi(l) \to \Phi(l) + \frac{1}{4\pi}\int \alpha(-\infty, l')\, d^2 l'\,.$$

"memory effect"

- test particle in free external field

$$\ddot{z}^b = \tfrac{q}{m} F^{bc}(z)\dot{z}_c\,, \quad z(0) = 0$$

$$F_{ab}(x) = -\frac{1}{2\pi}\int \Big\{ l_a \ddot{V}_b(x\cdot l, l) - l_b \ddot{V}_a(x\cdot l, l)\Big\}\, d^2 l$$

- infrared limit

$$A_\lambda(x) = \lambda^{-1} A(\lambda^{-1}x) \quad \lambda \to \infty$$

then $\qquad F_\lambda(x) = \lambda^{-2} F(\lambda^{-1}x)$

energy content $\qquad \mathcal{E}_\lambda = \lambda^{-1}\mathcal{E}$

– the field and its energy content tend to zero,
but the long-range tail is conserved

- In the limit $\dot{z} \to \dot{z}_{-\infty} = \dot{z}_{\infty} \equiv v$
- The only effect of scattering: trajectory shift by

$$\delta^b = -\frac{q}{2\pi m} \int \frac{l^b V^c(-\infty, l) - l^c V^b(-\infty, l)}{(v \cdot l)^2} \, d^2 l \; v_c$$

- Shift completely determined by tail of the elmg field
- Staruszkiewicz 1981: efect discovered for quantum particle in semiclassical approximation

matter radiation system

- Maxwell equations

$$\Box A(x) = 4\pi J(x)$$

- currents stabilize in past and future

$$J(\lambda x) \sim \lambda^{-3} x\, \rho(x) \quad \text{for} \quad x^2 > 0 \quad \text{and} \quad \lambda \to \infty\,,$$

- denote

$$V_J(s, l) = \int J(x)\, \delta(s - l \cdot x)\, d^4 x$$

- then $Q = l \cdot V_J(s, l)$ and

$$\lim_{R\to\infty} R A^{\mathrm{ret}}(x + Rl) = V_J(x \cdot l, l)$$
$$\lim_{R\to\infty} R A^{\mathrm{adv}}(x - Rl) = V_J(x \cdot l, l)$$

$$\lim_{R\to\infty} RA^{\mathrm{ret}}(x - Rl) = V_J(-\infty, l)$$
$$\lim_{R\to\infty} RA^{\mathrm{adv}}(x + Rl) = V_J(+\infty, l)$$

- for $A = A^{\mathrm{ret}} + A^{\mathrm{in}} = A^{\mathrm{adv}} + A^{\mathrm{out}}$ again

  $$\lim_{R\to\infty} RA(x + Rl) = V(x \cdot l, l)\,, \qquad \lim_{R\to\infty} RA(x - Rl) = V'(x \cdot l, l)$$

  but now

  $$V(+\infty, l) = V_J(+\infty, l)\,, \quad V'(-\infty, l) = V_J(-\infty, l)\,,$$

  – gauge invariant quantities in Lorenz class –
  and

  $$V(s, l) + V'(s, l) - V_J(s, l) = V'(+\infty, l) = V(-\infty, l)\,.$$

“matching property”

- in consequence

$$V'(+\infty, l) = V(-\infty, l)$$

- related to the “conservation of spacelike tail” $(y^2 < 0)$

$$\lim_{R\to\infty} RA(x + Ry) = \int V(-\infty, l)\delta(y \cdot l)d^2 l$$

- scattering $q_i', v_i' \longrightarrow q_j, v_j$, then

$$V_J(-\infty, l) = \sum_{i=1}^{n'} q_i' \frac{v_i'}{v' \cdot l}\,, \quad V_J(+\infty, l) = \sum_{i=1}^{n} q_i \frac{v_i}{v \cdot l}\,.$$

$$2\pi \lim_{\omega\to 0} \omega\big(a^{\mathrm{out}}(\omega l) - a^{\mathrm{in}}(\omega l)\big) = \sum_{i=1}^{n'} q_i' \frac{v_i'}{v' \cdot l} - \sum_{i=1}^{n} q_i \frac{v_i}{v \cdot l}$$

free infrared singular fields

- IR singular: $V(-\infty, l) = 2\pi \lim_{\omega \to 0} \omega a(\omega l) \neq 0$
- quantum theory: Fock space – only IR regular; coherent state representations – fixed $V(-\infty, l)$
- 'in' and 'out' cannot be simultaneously IR regular (or with the same IR tail)
- phenomenological approach – set a minimal threshold for photon energy, inclusive cross section etc. (YFS)
- dominating more fundamental approach – in addition to Coulomb field attach rigidly 'clouds' of radiation to charged particles so as to make rhs in the matching equation vanish
- if one wants to avoid this (charged particle carry their Coulomb fields), then one needs a larger free electromagnetic fields algebra (and its representations) including IR singular fields
- further constructions follow this idea

timelike infinity

- free Dirac field

$$\psi_{\rm free}(x) = \left(\tfrac{m}{2\pi}\right)^{3/2} \int e^{-im\, x\cdot v\, \gamma\cdot v}\, \gamma\cdot v\, f(v)\, d\mu(v)\,,$$

- timelike asymptote: for $\lambda \to \infty$

$$\psi_{\rm free}(\pm\lambda v) \;\sim\; \mp i\,\lambda^{-3/2} e^{\mp i(m\lambda+\pi/4)\gamma\cdot v} f(v)$$

- theorem For $\psi(x)$ in external potential in gauge

$$A_{\rm tr}(x) = A(x) - \partial S(x)\,, \quad S(x) \simeq \log\sqrt{x^2}\; x\cdot A(x) \quad \text{for} \quad x^2 \to \infty$$

there is $\qquad v\cdot A_{\rm tr}(\pm\lambda v) \sim \lambda^{-1-\varepsilon} \qquad$ and

$$\psi(\pm\lambda v) \;\sim\; \mp i\,\lambda^{-3/2} e^{\mp i(m\lambda+\pi/4)\gamma\cdot v} f_{\pm}(v) \qquad \text{for} \quad \lambda \to \infty\,.$$

- f, $f_{\pm}$ are gauge-invariant

invariant structures and Poincaré transformations

- symplectic form – admits IR singular fields

$$\{V_1, V_2\} = \frac{1}{4\pi} \int (\dot{V}_1 \cdot V_2 - \dot{V}_2 \cdot V_1)(s, l)\, ds d^2 l$$

- pre-Hilbert scalar product

$$(f_1, f_2) = \int \overline{f_1(v)} \gamma \cdot v\, f_2(v)\, d\mu(v)$$

- Poincaré generators

$$(r_a V)_c(s, l) = -l_a \dot{V}_c(s, l)\,, \quad (p_a f)(v) = m v_a \gamma \cdot v f(v)\,,$$

$$(n_{ab} V)_c(s, l) = -L_{ab} V_c(s, l) - g_{ca} V_b(s, l) + g_{cb} V_a(s, l)\,,$$

$$(m_{ab} f)(v) = i\Big(v_a \partial/\partial v^b - v_b \partial/\partial v^a + \tfrac{1}{4}[\gamma_a, \gamma_b]\Big) f(v)\,,$$

conserved quantities

- outgoing
$$P_a^{\rm out} = (f_+, p_a f_+) + \tfrac{1}{2}\{V, r_a V\}$$
$$M_{ab}^{\rm out} = (f_+, m_{ab} f_+) + \tfrac{1}{2}\{V, n_{ab} V\}$$
– equal to similar incoming
- transformation
$$g_+(v) = \exp\left(\frac{ie}{4\pi}\int \frac{\Phi^{\rm out}(l)}{(v\cdot l)^2}\, d^2 l\right) f_+(v)$$
gives
$$P_a^{\rm out} = (g_+, p_a g_+) + \tfrac{1}{2}\{V^{\rm out}, r_a V^{\rm out}\}\,,$$
$$M_{ab}^{\rm out} = (g_+, m_{ab} g_+) + \tfrac{1}{2}\{V^{\rm out}, n_{ab} V^{\rm out}\}\,.$$
- g_+ – particle together with its Coulomb field
- further construction with the use of fields with clear physical interpretation: total electromagnetic field V and charged particle surrounded by its Coulomb field, $g_+ \equiv g$

quantum algebra

- quantization: $g \to g^q$, $V \to V^q$
- smeared elements

$$\Psi(g) = (g, g^q)\,, \quad W(V) = \exp\big[-i\{V, V^q\}\big]$$

- $\Psi(g)$ adds a particle and its Coulomb field
- algebra

$$W(V_1)W(V_2) = e^{-\frac{i}{2}\{V_1, V_2\}} W(V_1 + V_2)\,,$$

$$W(V)^* = W(-V)\,, \quad W(0) = 1\,,$$

$$[\Psi(g_1), \Psi(g_2)]_+ = 0\,, \quad [\Psi(g_1), \Psi(g_2)^*]_+ = (g_1, g_2)1\,,$$

$$W(V)\Psi(g) = \Psi(S_\Phi g)W(V)$$

where

$$(S_\Phi g)(v) = \exp\left(i\frac{e}{4\pi}\int \frac{\Phi(l)}{(v\cdot l)^2}\, d^2 l\right) g(v)$$

representations

- positive energy representations act in $\mathcal{H} = \mathcal{H}_F \otimes \mathcal{H}_r$
- $\mathcal{H}_F$ – Fock space for fermions, $\Psi(g)$ act in standard way on $\mathcal{H}_F$, tensored with identity on $\mathcal{H}_r$
- $\mathcal{H}_r$ – representation space of a positive, regular representation $W_r(V)$ of the electromagnetic part of algebra
  then
  $$W(V) = e^{-i\{V, V^q(+\infty,.)\}} \otimes W_r(V)$$
  where
  $$V^q(+\infty, l) = \int n^q(v) \frac{e\,v}{v\cdot l}\, d\mu(v)\,, \quad n^q(v) = :\overline{g^q(v)}\gamma\cdot v g^q(v):$$

representations $W_r(V)$

- these representations are not unique
- physically natural class obtained as follows
- start with non-regular vacuum state

$$\hat{\omega}(W(V)) = \begin{cases} \exp\left[-\frac{1}{2}\int_{\omega\geqslant 0} |\tilde{\dot{V}}(\omega, l)|^2 \frac{d\omega}{\omega} d^2 l\right] & \tilde{\dot{V}}(0, l) = 0 \\ 0 & \tilde{\dot{V}}(0, l) \neq 0 \end{cases}$$

  GNS construction → direct sum of coherent sectors labeled by the shape of the long-range tail

- we need regular representation: integrate these coherent sectors with some gaussian measure.
  this gives a class of regular, positive energy representations, in which there is no vacuum.
- photons may be added and subtracted from the background.

back to classical theory: other gauges

- $\hat{A} = A + \partial\Lambda$, but we keep restriction

$$\lim_{R\to\infty} R\hat{A}_b(st + Rl) = \hat{V}_b(s, l)\,,$$

- it follows

$$\lim_{R\to\infty} R(\partial_b\Lambda)(st + Rl) = \hat{V}_b(s, l) - V_b(s, l)$$

(scaling $t \cdot l = 1$)

- Therefore

$$\Lambda(st + Rl) = \varepsilon^+(l) + \frac{\beta_t(s,l)}{R} + o(R^{-1})\,.$$

and

$$(\partial_b\Lambda)(st + Rl) = \frac{1}{R}\Big[V_b^+(l) + l_b\dot{\gamma}(s,l)\Big] + o(R^{-1})\,.$$

where

$$l_a V_b^+(l) - l_b V_a^+(l) = L_{ab}\varepsilon^+(l)$$

$$\varepsilon^+(l) = \frac{1}{4\pi}\int \frac{l\cdot V^+(l')}{l\cdot l'}\,d^2l'$$

$$\dot{\gamma}(s,l) = \dot{\beta}_t(s,l) - t\cdot V^+$$

- Then
- ε^+ and $\dot{\gamma}$ independent of t, and

$$\hat{V}_b(s,l) = V_b(s,l) + V_b^+(l) + l_b\dot{\gamma}(s,l)$$

- up to Lorenz gauge:

$$\hat{V}_b(s,l) = V_b(s,l) + V_b^+(l)$$

  – large gauge transformation (LGT)

LGT is not a gauge symmetry of the asymptotic structure:

- No equation in the bulk of Minkowski space
- Symplectic structure and angular momentum change:

$$\{\hat{V}_1, \hat{V}_2\} = \{V_1, V_2\} + \frac{1}{4\pi}\int \left[V_1^+(l)\cdot \Delta V_2(l) - V_2^+(l)\cdot \Delta V_1(l)\right] d^2 l\,,$$

$$\Delta V_i(l) = V_i(-\infty, l) - V_i(+\infty, l)\,,$$

$$\tfrac{1}{2}\{\hat{V}, n_{ab}\hat{V}\} = \tfrac{1}{2}\{V, n_{ab}V\} + \{V, n_{ab}V^+\}\,.$$

LGT is a physical transformation of the asymptotic state

- decompose $V(s, l)$ into

$$V(+\infty, l) = \int \frac{v}{v \cdot l} \rho_+(v) \, d\mu(v)$$

$$V(s, l) - V(+\infty, l) = V^{\text{out}}(s, l)$$

then

$$\hat{V}(+\infty, l) = V(+\infty, l) + V^+(l)$$

$$\hat{V}(s, l) - \hat{V}(+\infty, l) = V^{\text{out}}(s, l)$$

- Theorem

  For almost all V^+ (in appropriate sense) there exists the representation

$$V^+(l) = \int \frac{v}{v \cdot l}\, \rho(v)\, d\mu(v)\,,$$

  where $\rho(v)$ is a smooth function of compact support on the unit hyperboloid, and such that $\int \rho(v)\, d\mu(v) = 0$.

- Therefore

$$\hat{V}(+\infty, l) = \int \frac{v}{v \cdot l}\, (\rho^+(v) + \rho(v))\, d\mu(v)\,,$$

- LGT adds an outgoing asymptotic, uncharged current to the state

quantum variables at spacial infinity

- classically these are long-range tails labels $V(-\infty, l)$
- quantum version $V^q(-\infty, l)$ needs better specification to make sense
- it needs smearing in l, thus we go to

  $$U(V^\varepsilon) = \exp\left\{\frac{i}{4\pi}\int V^\varepsilon(l)\cdot V^q(-\infty, l)\, d^2 l\right\},$$

  where $V^\varepsilon(l)$ is a vector test function, homogeneous of degree -1, and such that

  $$l_a V^\varepsilon_b(l) - l_b V^\varepsilon_a(l) = L_{ab}\varepsilon(l)$$

  for some $\varepsilon(l)$ homogeneous of degree 0.
- despite smearing such elements are not defined in the algebra because of the formal value $-\infty$

- we split

$$V^q(-\infty, l) = V^q(+\infty, l) + [V^q(-\infty, l) - V^q(+\infty, l)] .$$

- first part due to the Coulomb fields of outgoing particles in our representation has a well defined meaning

$$U^{\mathrm{Coul}}(V^\varepsilon) = \exp\left\{\frac{i}{4\pi}\int V^\varepsilon(l)\cdot V^q(+\infty, l)\, d^2 l\right\}\otimes \mathrm{id}$$

$$= \exp\left\{\frac{ie}{4\pi}\int n^q(v)\int \frac{v\cdot V^\varepsilon(l)}{v\cdot l}\, d^2 l\, d\mu(v)\right\}\otimes \mathrm{id}$$

$$= \exp\left\{\frac{ie}{4\pi}\int n^q(v)\int \frac{\varepsilon(l)\, d^2 l}{(v\cdot l)^2}\, d\mu(v)\right\}\otimes \mathrm{id}$$

- second part due to free 'out'field
  we expect to obtain the corresponding operator by some limiting at the level of representation
- let $\widetilde{h}(\omega, l)$ be a smooth, fast vanishing for $|\omega| \to \infty$, such that $\widetilde{h}(\lambda^{-1}\omega, \lambda l) = \widetilde{h}(\omega, l)$ $(\lambda > 0)$, and $\widetilde{h}(0, l) = 1$ denote

$$\widetilde{V}^{\varepsilon}_{\beta}(\omega, l) = \frac{-|\omega|^{\beta-1}\widetilde{h}(\omega, l)}{2\int |u|^{\beta-1}|\widetilde{h}(u, l)|^2 du}\, V^{\varepsilon}(l)\,.$$

  then for classical function $V(s, l)$:

$$\lim_{\beta\searrow 0}\{V^{\varepsilon}_{\beta}, V\} = -\frac{1}{4\pi}\int V^{\varepsilon}(l)\cdot[V(-\infty, l) - V(+\infty, l)]\, d^2 l\,.$$

- theorem strong limit exists:

$$U^{\mathrm{free}}(V^{\varepsilon}) = \mathrm{id}\otimes \lim_{\beta\searrow 0} W_r(V^{\varepsilon}_{\beta})$$

quantum variables at infinity

- Coulomb + free variables

$$U(V^{\varepsilon}) = U^{\mathrm{Coul}}(V^{\varepsilon})\, U^{\mathrm{free}}(V^{\varepsilon})$$

- commutation relations

$$U(V^{\varepsilon})W(V_1) = e^{\frac{i}{4\pi}\int V^{\varepsilon}(l)\cdot V_1(-\infty, l)\, d^2 l}\, W(V_1)\, U(V^{\varepsilon})$$
$$U(V^{\varepsilon})\Psi(g_1) = \Psi(S_{\varepsilon}g_1)U(V^{\varepsilon})\,,$$

- variables at infinity are conserved, so if the scattering operator interpolates between 'in' and 'out' representations, then it should hold

$$[U(V^{\varepsilon}), S] = 0$$

transformation

$$U(V^+)\exp[-i\{V_1, V^q\}]U(V^+)^* = U^{\mathrm{free}}(V^+)\dots U^{\mathrm{free}}(V^+)^*$$
$$= \exp[-i\{V_1, V^q + V^+\}]$$

$$U(V^+)\Psi(g_1)U(V^+)^* = U^{\mathrm{Coul}}(V^+)\dots U^{\mathrm{Coul}}(V^+)^*$$
$$= \Psi(S_{\varepsilon^+} g_1)\,,$$

- second-kind-gauge invariant quantities nontrivially transformed
- addition of the asymptote V^+ of the Coulomb field of a classical, charge-free current
- transformation of $\Psi(g_1)$ is naturally interpreted as the infrared-limit transformation

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