QUANTUM BACKREACTION (CASIMIR) EFFECT WITHOUT INFINITIES ALGEBRAIC ANALYSIS

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- Casimir effect, in most general terms, is the backreaction of a quantum system responding to an adiabatic change of external conditions. This backreaction is expected to be quantitatively measured by a change in the expectation value of a certain energy observable of the system.
- However, for this concept to be applicable, the system has to retain its identity in the process. Most prevailing tendencies in the analysis of the effect seem to overlook this question.

- In general, a quantum theory is defined by an algebra of observables, whose representations by operators in a Hilbert space define concrete physical systems described by the theory. A quantum system retains its identity if both the algebra as well as its representation do not change.
- I shall discuss the resulting restrictions for admissible models of changing external conditions. These ideas are applied to quantum field models. No infinities arise, if the algebraic demands are respected.

1. QUANTUM SYSTEM UNDER EXTERNAL CONDITIONS

2. A CLASS OF QUASI-FREE SYSTEMS

3. ADMISSIBILITY OF MODELS

4. TWO MODELS FOR PARALLEL PLANES

5. SCALING

6. SUMMARY AND OUTLOOK

SYSTEM Q-M

- Q – relatively simple quantum system (e.g. a quantum field)
- M – complex macroscopic system (say, conducting plates) with collective effective coordinates a
- Full closed theory of Q-M out of reach
- Approximation:
  M is 'heavy' – characterized by very large inertia; thus:
- variables a are classical
- changes of a are adiabatic

Isolated system Q (HP)

- Basic quantum variables at a fixed time form an abstract algebra $\mathcal{A}$, e.g. CCR algebra.
- Algebra is represented by operators in a Hilbert space $\mathcal{H}$:

  $$\pi : \mathcal{A} \mapsto \pi(\mathcal{A}) \,, \quad A \mapsto \pi(A);$$

  Density operators in $\mathcal{H}$ represent states of the system Q.
- Intrinsic dynamics of Q defined by an automorphism of $\mathcal{A}$:

  $$\alpha_t : \mathcal{A} \mapsto \mathcal{A} \,, \quad A \mapsto \alpha_t A$$

  implemented by a unitary evolution in the Hilbert space $\mathcal{H}$:

  $$\pi(\alpha_t A) = U(t)\pi(A)U(t)^* \,, \quad U(t) = \exp(itH) \,,$$

  where H – the energy operator of the system, with nonnegative spectrum and a ground state, represented by a unit eigenvector; energy may be normalized to be zero in that state.

Introducing M

- Add part M into the system: characterized by classical variables a; no quantum degrees added.
- System Q should retain its identity: algebra $\mathcal{A}$ must remain unaffected.
- States to be considered must be physically comparable: the representation π of $\mathcal{A}$ must remain unaffected.

Dynamics of Q with frozen M

- Degrees *a* frozen – system Q still a closed system in interaction with conditions created by M; for each a evolution: an automorphism of $\mathcal{A}$:

$$\alpha_{at} : \mathcal{A} \mapsto \mathcal{A}\,, \quad A \mapsto \alpha_{at} A\,.$$

- Evolution implemented in representation π: for each a

$$\pi(\alpha_{at} A) = U_a(t)\pi(A)U_a(t)^*\,, \quad U_a(t) = \exp(itH_a)\,.$$

- For each a the generator H_a defined by this up to:

$$H_a \to H_a + \lambda_a \,\mathrm{id}\,,$$

where λ_a is any real function of parameters a.

Coupled system Q-M (SP)

- Unitary evolution of Q in SP (Q not closed – evolution on algebraic level: to restrictive).
- Suppose that $a(t)$ is known as a 'slow' function of time (system M is 'heavy'). Adiabatic approximation with initial ($t = 0$) eigenstate of $H_{a(0)}$:

$$\psi(t) = e^{i\varphi(t)}\psi_{a(t)} ,$$

  where $H_a\psi_a = E_a\psi_a$ and $\varphi(t)$ is a real function depending functionally on E_a and ψ_a.
- Evolution of expectation value of an observable B given by

$$\langle B\rangle_t = (\psi_{a(t)}, B\,\psi_{a(t)}) ,$$

Backreaction – determination of $a(t)$

- Intrinsic energy stored in part Q of the system represented by H, which in the coupled system is not a constant of motion any more; its expectation value

$$\mathcal{E}_a := (\psi_a, H\psi_a)\,,$$

  depends on time through variables a.
- Changes in $\mathcal{E}_a$ correspond to the energy which has been transferred from Q to the rest of the system, which (with the suppression of all microscopic details of M) is described by the variables a. Thus $\mathcal{E}_a$ plays the role of a potential energy with respect to these variables. We assume that the rest of the total energy of the coupled system is supplied by the kinetic energy of M, thus we obtain a potential system, with the generalized force given by

$$\mathcal{F}_a = -\frac{\partial \mathcal{E}_a}{\partial a}\,.$$

Classical system

- Symplectic space (phase space):
  $\mathcal{L} \subset \mathcal{R} \oplus \mathcal{R} \ni V \equiv (v \oplus u)$, $\mathcal{R}$ – real Hilbert space
  symplectic form

$$\sigma(V_1, V_2) = (v_2, u_1) - (v_1, u_2)$$

- Hamiltonian function

$$\mathcal{H}(v, u) = \frac{1}{2}[(u, u) + (hv, hv)]\,,$$

  h – positive selfadjoint operator on $\mathcal{R}$,
- Symplectic evolution

$$T_t(v \oplus u) = \big(\cos(ht)v + \sin(ht)h^{-1}u\big) \oplus \big(-\sin(ht)hv + \cos(ht)u\big)\,.$$

- Denote

$$V'(V) = (v', u) + (u', v)\,,$$

  then

$$(T_t V')(V) = V'(T_t V)\,.$$

Quantum system

- 'Quantization': $V'(V) \to \Phi(V)$ – algebraic elements satisfying CCR

$$[\Phi(V_1), \Phi(V_2)] = i\sigma(V_1, V_2)\,\mathrm{id}\,, \quad V \in \mathcal{L}$$

evolution

$$\alpha_t(\Phi(V)) = \Phi(T_t V)$$

- Vacuum representation $\Phi(V) \to \Phi_0(V)$ – operators in a Fock space; rep. defined by demands:

$$\Phi_0(T_t V) = U(t)\Phi_0(V)U^*(t)\,, \quad U(t) = \exp(itH)\,.$$

H – positive, with ground state vector Ω

Examples

- Free massless scalar field – initial value formulation:

$$\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^3)\,, \quad h = \sqrt{-\Delta}\,, \quad \mathcal{L} = \mathcal{D}_{\mathbb{R}}(\mathbb{R}^3) \oplus \mathcal{D}_{\mathbb{R}}(\mathbb{R}^3)$$

- Scalar field with boundary conditions on surfaces coordinated by parameters a:

$$\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^3)\,,\ [h^B_a]^2 = -\Delta_{\mathrm{b.c.}}\,,\ \mathcal{L}^B_a = \mathcal{D}_{\mathcal{R}}(h^B_a) \oplus \mathcal{D}_{\mathcal{R}}([h^B_a]^{-1/2})$$

- 'Momentum'-regularized boundary conditions: $h_a = f(h, h^B_a)$, such that $h_a \simeq h^B_a$ for small momentum transfer, and $h_a \simeq h$ for large momentum transfer
- Scalar field with external static interaction depending on parameters a: $h^2_a = -\Delta + V_a$, V_a – perturbation

Examples – quantization

- Algebra CCR

$$[\Phi(V_1), \Phi(V_2)] = i\sigma(V_1, V_2)\,\mathrm{id}\,, \quad V \in \mathcal{L}_a$$

evolution

$$\alpha_{at}(\Phi(V)) = \Phi(T_{at}V)$$

where T_{at} is defined by h_a

- Ground state representation $\Phi(V) \to \Phi_a(V)$ – operators in a Fock space; rep. defined by demands:

$$\Phi_a(T_t V) = U_a(t)\Phi_a(V)U_a^*(t)\,, \quad U_a(t) = \exp(itH_a)\,.$$

H_a – positive, with ground state vector Ω_a

Conditions

- Stability of algebras: $\mathcal{L} = \mathcal{L}_a$
  Not satisfied in the sharp boundary conditions case!
  (and no way to satisfy the condition by any extension of symplectic spaces)
- Relation between representations: when stability of algebras is ensured then annihilation/creation operators of representations determined by h and h_a are related by a Bogoljubov transformation

$$a_a(f) = a(T_a f) + a^*(S_a f)\,,\ a_a^*(f) = a^*(T_a f) + a(S_a f)$$

with T_a linear and S_a antilinear, determined by h and h_a.

Conditions

- Representations are equivalent iff

  $$\mathcal{N}_a \equiv \mathrm{Tr}[S_a S_a^*] = \tfrac{1}{4}\,\mathrm{Tr}[h^{-1/2}(h_a - h)h_a^{-1}(h_a - h)h^{-1/2}] < \infty$$

  Then $\mathcal{N}_a = (\Omega_a, N\Omega_a)$, N – particle (or ‘excitation’) number
- Casimir energy for ground state

  $$\mathcal{E}_a = (\Omega_a, H\Omega_a) = \tfrac{1}{4}\,\mathrm{Tr}[(h_a - h)h_a^{-1}(h_a - h)]$$

  If this happened to be infinite, this would have a perfectly legitimate physical meaning: creation of Ω_a, although theoretically possible, needs infinite amount of energy.

Planar symmetry

- $\mathcal{R} = L^2_{\mathbb{R}}(\mathbb{R}^2) \otimes L^2_{\mathbb{R}}(\mathbb{R})$,

$$h^2 = (h_\perp \otimes \mathrm{id})^2 + (\mathrm{id} \otimes h_z)^2\,, \quad h_a^2 = (h_\perp \otimes \mathrm{id})^2 + (\mathrm{id} \otimes h_{za})^2$$

$$h_\perp^2 = -\Delta_\perp\,, \quad h_z^2 = -\partial_z^2$$

- h_{za} will model parallel planes at a distance a
- $\mathcal{N}_a$ and $\mathcal{E}_a$ must be normalized to quantities per unit area of planes, n_a and ε_a resp.

Two models

- 1. Transparency to large momenta:

$$h_{za} = h_z + G(h_z)\big[F(h^B_{za}) - F(h_z)\big]G(h_z)\,,$$

h^B_{za} – sharp Dirichlet or Neumann bound. cond. at points separated by a,
$F, G(p) \to 0$ $(p \to \infty)$, $\quad F(p) = p$, $G(p) = 1$ $(p \leq p_0)$

- 2. Nonlocality control:

$$h^2_{za} = h^2_z + V_a\,, \quad V_a(z,z') = g(z-b)\overline{g(z'-b)} + g(z+b)\overline{g(z'+b)}$$

g of compact support, even (D) or odd (N) $(b = a/2)$

- Both classes of models meet the admissibility criteria, so the Casimir energy per area ε_a is well defined and finite for all a

Approximating boundary conditions

- For each model – a one parameter ($\lambda \in (0, 1\rangle$) family of rescaled models, such that for fixed a and $\lambda \to 0$ the sharp boundary conditions are recovered
- The Casimir energy per area scales:

$$\varepsilon_{a,\lambda} = \lambda^{-3} \varepsilon_{a/\lambda}$$

Thus to find scaling behavior of $\varepsilon_{a,\lambda}$ for scaled models: expand ε_a in powers of $1/a$ up to third order

Expanding ε_a

- Casimir energy ε_a given by a complex integral expression depending functionally on functions defining the models and on a
- Expansion

$$\varepsilon_a = \varepsilon_\infty + \frac{\gamma}{a^2} - \frac{\pi^2}{1440 a^3} + \text{higher terms}$$

ε_∞ energy needed to produce field configuration around two independent plates
γ $= 0$ for Dirichlet case, and
model dependent for Neumann case
– contribution from inside the walls
a^{-3}-term – universal for large class of models

- Models for conducting plates and electromagnetic field – sum of the D and N terms

Summary and outlook

- Proper formulation of the problem removes the usual sources of infinities
- Casimir energy defined as expectation value of one and the same observable for all modifications of external conditions in question
- Models considered very simplified, but showing the strength of the formulation
- Large area for further research: more refined models of external bodies, thermal states, different geometries,...

Bibliography

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